Journal of Statistical Physics, Vol. 102, Nos. 5/6, 2001

Ising-Type and Other Transitions in One-Dimensional Coupled Map Lattices with Sign Symmetry

C. Boldrighini,[1] L. A. Bunimovich,[2] G. Cosimi,[3] S. Frigio,[3] and A. Pellegrinotti[4]

Received February 28, 2000

We consider a one-dimensional lattice of expanding antisymmetric maps $[-1, 1] \to [-1, 1]$ with nearest neighbor diffusive coupling. For such systems it is known that if the coupling parameter ε is small there is unique stationary (in time) state, which is chaotic in space-time. A disputed question is whether such systems can exhibit Ising-type phase transitions as ε grows beyond some critical value ε_c. We present results from computer experiments which give definite indication that such a transition takes place: the mean square magnetization appears to diverge as ε approaches some critical value, with a critical exponent around 0.9. We also study other properties of the coupled map system.

KEY WORDS: Chaotic systems; coupled map lattices; phase transitions; Ising-type transitions.

1. INTRODUCTION

Lattice dynamical systems (LDC) proved to be useful models in the study of many physical, chemical, hydrodynamical and biological systems.[1, 2] The studies of LDS allowed to make some serious steps towards a better understanding of such phenomena as space-time chaos, space and space-time intermittency and pattern formation. An important step was the exact

[1] Dipartimento di Matematica G. Castelnuovo, Università "La Sapienza," Piazzale Aldo Moro 5, 00185 Roma, Italy.

[2] Southeast Applied Analysis Center; School of Mathematics, Georgia Insitute of Technology, Atlanta, Georgia 30332.

[3] Dipartimento di Matematica e Fisica, Università di Camerino, Via Madonna delle Carceri 9, 62032 Camerino, Italy.

[4] Dipartimento di Matematica, Universitá degli studi di Roma Tre, Largo San Leonardo Murialdo 1, 00146 Roma, Italy.

0022-4715/01/0300-1271$19.50/0

definition of space-time chaos (space-time mixing),[3] a property which has been proved to be shared by some classes of LDS.[3–6]

The thermodynamic formalism developed in ref. 3 and later extended in refs. 4–8 allowed to build a bridge between hyperbolic LDS and the lattice spin systems of statistical mechanics. Especially, it was proposed in ref. 3 that the existence of a unique invariant Gibbs measure with absolutely continuous finite-dimensional marginals in lattices of weakly interacting (uniformly) hyperbolic local maps corresponds to the absence of phase transitions for high temperatures in the corresponding lattice spin system of statistical mechanics. Thus it has been suggested in ref. 3 that the phase transitions in those systems of statistical mechanics, which may appear in the range of low temperatures, could be interpreted as appearance of coherent structures from the state of space-time chaos when the strength of spatial interaction (order parameter) exceeds some critical value.

Since then many publications dealing with this exciting problem have appeared (see e.g., refs. 9–11). The great majority of the papers are based on computer experiments. There are however a few papers[15, 17] in which some rigorous results are proved, which confirm the phenomenology suggested in ref. 3.

It is worth to mention, in passing, that the word "phase transition" has now become fashionable and is widely used instead of the "honest" word "bifurcation," even when no thermodynamic formalism has been constructed. One should actually be very careful in using the term "phase transitions" for dynamical systems. We think that the relevant approach is that suggested in ref. 15, which we outline below.

"Phase Transitions" deal with equilibrium states for a system of statistical mechanics. Therefore, the corresponding notion for dynamical systems must deal with invariant measures, with respect to the dynamics. However, a dynamical system with chaotic behavior usually has infinitely many (actually a continuum) of invariant measures. Therefore, the crucial question is to determine which invariant measures are relevant. The thermodynamic formalism developed for hyperbolic dynamical systems[18, 19] qualifies such measures as Gibbs measures. It is now widely accepted that the SRB measures form a relevant class of invariant measures for hyperbolic dynamical systems. Accordingly, the recent paper[17] suggests that the notion of phase transition in spatially extended dynamical systems should deal with SRB measures. There are at least two objections to such approach. First of all, in principle, phase transitions may occur not only in hyperbolic dynamical systems, whereas the notion of SRB measures is defined only for hyperbolic systems. Secondly, at the point of phase transition a dynamical system may loose hyperbolicity, so that the notion of SRB measures would lose its sense at the critical point.

The approach suggested in ref. 15, instead, does not have such deficiencies. Moreover, it contains the one suggested in ref. 17 as a special case, when the notion of SRB measure has sense before as well as after the phase transition. This approach deals with the class of the stable invariant measures, which are limiting (under the dynamics) points for some class of natural non-equilibrium (non-invariant) measures. Such class of invariant measures for extended dynamical systems is formed by the measures which have finite-dimensional absolutely continuous marginals.[3, 15] The approach suggested in ref. 15, besides its generality and its advantages for the mathematical study of spatially extended dynamical systems, is also very natural for numerical studies and the interpretation of the outcoming results. Indeed, in numerical studies, which always deal with finitely extended systems, one usually considers the evolution of a phase volume (Lebesgue measure).

Therefore in the present paper we follow the approach of ref. 15, which states that there is a phase transition at the critical value p_{cr}, of the order parameter p if for $p < p_{cr}$ all measures in some class of "natural" non-invariant measures $\mathscr{M}$ converge to a measure m, whereas for $p > p_{cr}$ a massive piece (at least) of the space of these measures begins to converge instead to some other measure(s), $(m_1,..., m_k)$, $k \geqslant 1$, which do not (all) coincide with m.

Since a possible relation between the appearance of coherent structures in extended dynamical systems and phase transitions in systems of statistical mechanics was discovered, there were several attempts to find out a model of extended dynamical system which exhibit transitions similar to the most popular phase transitions in statistical physics, those of Ising model. There was first a result by Miller and Huse,[13] which considered a two-dimensional coupled map lattice, for which they claimed to obtain a phase transition similar to the one in Ising models. However, the critical indices for this coupled map lattice turned out to be different from the ones in Ising model.[16] The same local maps with diffusion coupling but with sequential, rather than simultaneous, update, allowed to obtain, for the corresponding coupled map lattice, the same critical indices as for the Ising model.[16]

Another conjecture suggested in refs. 3 and 8 claims that phase transitions in Lattice Dynamical Systems may occur already in one dimension. Indeed, time dynamics provides the second dimension, which adds to the spatial dimension, in the corresponding system of statistical mechanics generated by the thermodynamic formalism. Since that time the existence of phase transitions in one dimensional lattice systems has been demonstrated numerically and analytically.[9–17] However, the most convincing result to show the correctness of this claim would be an example of one-dimensional lattice dynamical system, which shows a phase transition similar to those of Ising models.

In the paper[14] we proposed that Ising-type transitions can take place in one-dimensional lattices of coupled "chaotic" maps, as the diffusive coupling constant grows away from 0. We argued that the critical value of the diffusive parameter for which such transition takes place is determined by a balance between local entropy production and coupling, which is expressed by the behavior of the Lyapunov dimension, namely by a dip in the plot of the Lyapunov dimension versus coupling strength, in the proximity of which the critical value should be found.

We consider here a new local map, which can be viewed as an "improved version" of the local map in the paper.[14] In fact the CML studied in that paper does not show an Ising-type transition. An *a priori* guess of the critical point is obtained as in ref. 14 by looking at the plot of the Lyapunov dimension vs the diffusive coupling constant ε.

For the new map we get a clear divergent behavior of the square magnetization near the critical point, and we are able to produce a fairly precise value for the critical exponent. Our results are, as we believe, definite evidence, as much as it is possible with computer experiments, of the occurrence of Ising-type transitions in one-dimensional CML maps.

We also present a thorough investigation of the behavior of the system as the coupling constant ε varies from 0 to values exceeding the critical value $\varepsilon_c \approx 0.58$. We find evidence of a variety of behaviors, which shows once more how rich the phenomenology simulated by CML systems can be.

We use throughout periodic boundary conditions.

2. THE MAP AND ITS PROPERTIES

The map $f: [-1, 1] \to [-1, 1]$ is continuous, antisymmetric and piecewise linear. Its analytic expression on the positive semi-interval is:

$$f(x) = \begin{cases} 5x & x \in [0, 0.2] \\ -\frac{8}{3}x + \frac{4.6}{3} & x \in [0.2, 0.5] \\ \frac{8}{3}x - \frac{3.4}{3} & x \in [0.5, 0.8] \\ -\frac{11}{2}x + 5.4 & x \in [0.8, 1] \end{cases} \tag{1.1}$$

As the slope is always larger than 1, the map is expanding. A plot of the map is given in Fig. 1.

We consider CML's on the one-dimensional lattice generated by a diffusion-type coupling of local (point) dynamical systems given by the above map, with periodic boundary conditions. The phase space of our CML is

$$\Omega_N = \{x = \{x_j \in [-1, 1] : j \in \mathbb{Z}_N\}\} \tag{1.2}$$

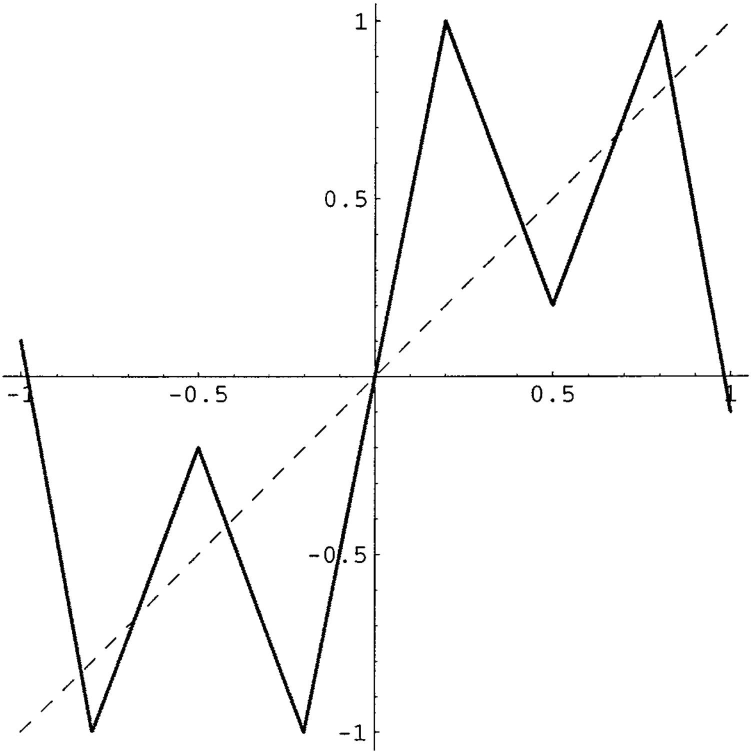

Fig. 1. Plot of the map.

where $\mathbb{Z}_N = \mathbb{Z}/(N\mathbb{Z})$ denotes the integers modulo N. On Ω_N we consider the maps Φ_ε and F with values in Ω_N and defined as follows:

$$(\Phi_\varepsilon x)_j = (1-\varepsilon)\, x_j + \varepsilon \sum_{k \in \mathbb{Z}_N} a_{|j-k|} x_k \tag{1.3a}$$

Here $0 \leqslant \varepsilon \leqslant 1$ and $\sum_j a_{|j|} = 1$. The map F is defined as

$$F_k(x) = f(x_k)$$

and the dynamics of the coupled map lattice is given by the composition

$$H_\varepsilon^N = \Phi_\varepsilon \circ F \tag{1.3b}$$

which maps Ω_N into itself. H_ε represents the subsequent action of the local map F and the coupling Φ_ε on Ω_N. Component-wise:

$$(H_\varepsilon^N x)_k = (1-\varepsilon)\, f(x_k) + \varepsilon \sum_{j \in \mathbb{Z}_N} a_{|j-k|} f(x_j) \tag{1.3c}$$

We consider only nearest neighbour diffusive coupling, for which (1.3a) is specified as

$$(\Phi_\varepsilon x)_k = (1-\varepsilon)\, x_k + \frac{\varepsilon}{2}(x_{k-1} + x_{k+1}) \tag{1.4}$$

It is worth to mention that practically all numerical studies of CML's were done for nearest neighbor coupling.

In accordance to a current jargon, a local variable x_j will sometimes be called "spin."

One can see form the plot in Fig. 1 that, in addition to the origin, the map f has three positive unstable fixed points, and three negative ones, antisymmetric to them. Moreover if N is even we have a fixed point of H_ε^N of space period two and of the type ..., $+a$, $-a$, $+a$, $-a$,..., if a is a solution of the equation

$$(1-2\varepsilon)\, f(a) = a$$

In addition to the solution $a = 0$, which is present for all $\varepsilon \in [0, 1]$, we have other solutions, which come, by antisymmetry, in pairs: three pairs of solutions for $0 < \varepsilon < 0.1$, and only one pair for $0.1 \leqslant \varepsilon < 0.4$. The smallest positive solution, which is there in the whole range $0 \leqslant \varepsilon < 0.4$ is

$$a_1(\varepsilon) = \frac{4.6 - 9.2\varepsilon}{11 - 16\varepsilon}$$

The other solutions are

$$a_2(\varepsilon) = \frac{3.4 - 6.8\varepsilon}{5 - 16\varepsilon}, \qquad a_3(\varepsilon) = \frac{5.4 - 10.8\varepsilon}{6.5 - 11\varepsilon}$$

All solutions are unstable in the whole range of ε.

The time evolution on Ω_N with initial condition $\bar{x} = \{\bar{x}_k : k \in \mathbb{Z}_N\}$ is the sequence of vectors: $\{x(t) : t = 0, 1,...\}$ with $x_k(0) = \bar{x}_k$ and $x_k(t+1) = (H_\varepsilon^N x(t))_k$.

For $\varepsilon = \frac{1}{2}$ it is easy to check that the system has a "first integral:" the phase point moves on the submanifold of equation $\sum_{k \text{ odd}} x_k(t) = \sum_{k \text{ even}} x_k(t)$. This may explain why correlations increase near this point.

3. BEHAVIOR FOR SMALL COUPLING

Throughout the paper N, the number of the space points, will be called "length" of the periodic chain, and R will denote the sample size, i.e., the number of independent CML systems with length N that are simulated. Unless otherwise stated the initial values of the spins $\{x_k(0) : k \in \mathbb{Z}_N\}$ are taken at random, with uniform distribution on $[-1, 1]$, independently for each k and for each sample.

For smooth maps, if the coupling constant ε is small we know that for the infinite system space-time chaos holds, i.e., there is an invariant measure which is mixing w.r.to space-time shifts.[3–6] The result is believed to hold also for piecewise smooth maps such as the one we consider here.

Already for values of ε of the order of 10^{-2} some kind of space structure appears. Namely, for N even, after some transient time, the signs of the spins settle on a periodic space pattern of period 2, constant in time (i.e., of time period 1). (For N odd there is a single defect.) This is connected to the presence of the unstable space periodic fixed points discussed in the paragraph above. In fact, by getting close to one of those solutions or oscillating among them, the system reaches eventually the region where neighboring spins have opposite signs, which appears to be stable in the given ε range. More precisely, for large t neighboring signs are opposite and there is an interval $I = (x_-, x_+)$, depending on ε, with $0 < x_- < 0.04$, $5.2/5.5 < x_+ < x_* = 5.4/5.5$, such that for all k, $|x_k(t)| \in I$. (Here x_* is the 0 of the map, i.e., $f(x_*) = 0$, and 0.04 and 5.2/5.5 are the points at which the map has the same value 0.2 of the central local minimum.) As $(1-\varepsilon)$ $\inf_{x \in I} f(x) > \varepsilon$, one can see that if neighboring signs are opposite at time t, then, for all k, sign $x_k(t) =$ sign $x_k(t+1)$.

What happens can actually be made clear by some simple rigorous analysis. Let $m(\varepsilon) = \inf\{f(x_-), f(x_+)\} = \inf_{x \in I} f(x)$. The stability of the period 2 pattern for the signs holds if the following inequalities are true

$$(1-\varepsilon)\, m(\varepsilon) \geqslant x_- + \varepsilon, \qquad x_* - x_+ \leqslant \varepsilon m(\varepsilon) + x_* - 1 + \varepsilon$$

In the second inequality we can replace $m(\varepsilon)$ by $f(x_+)$, computed by the expression on the last line of formula (1.1). This gives the condition $0 \leqslant x_* - x_+ \leqslant (\varepsilon - \varepsilon_0)/(1 - 5.5\varepsilon)$, with $\varepsilon_0 = 1 - x_* = 0.0\overline{18}$, i.e., we find a threshold $\varepsilon \geqslant \varepsilon_0$, very close to the observed one. By some tedious computations one can actually find points $x_\pm$, depending on ε, for which the

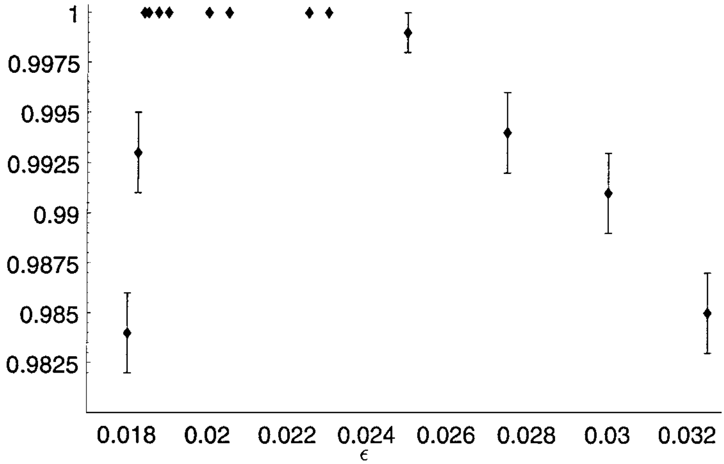

Fig. 2. Fraction of sign changes vs ε. $N = 400$, $R = 10$.

inequalities above hold for $\varepsilon \geqslant \varepsilon_1 := 0.4/17.5$. As the conditions stated are only sufficient, there is no contradiction with the data below.

Figure 2 shows the behavior in the range $0.018 \leqslant \varepsilon \leqslant 0.0325$ of the fraction of sign changes between neighboring spins at fixed time (i.e., number of such changes over N), for $N = 400$ and a sample of $R = 10$ independent cases. Error bars correspond to one standard error. The data refer to the stationary state, after a transient time of 8×10^7 units. In the range of ε where the fraction of the sign changes is 1, once the stationary regime is established, the sign of each spin remains constant in time, as predicted by the analysis above, though the spin itself varies over the whole interval $\pm I$ described above.

It is well known[1, 2, 9, 10, 12, 15] that in the range of spatial interactions above space-time chaos (small ε) stable oscillations often appear with period two in space and in time However the local maps considered in those papers always have only two branches of monotonicity. Our local map with many monotonicity branches rather produces several solutions with space-time periods two-one and two-two which all are unstable. So in this range of spatial interactions we have instead large variations in time of the values of local variables, with locally fixed signs, as described above.

As ε grows the periodic sign structrure breaks down, and "islands" of positive and negative spins begin to appear, their size growing with ε. In the whole range $0 \leqslant \varepsilon < \varepsilon_c$, where the critical ε, as explained below, is $\varepsilon_c \approx 0.58$, we have evidence of a unique stationary state (invariant measure) which is space homogeneous and attracting for all initial data, except for the region of spin locking just described where we have two measures which go into each other by a space shift.

4. BEHAVIOR NEAR THE CRITICAL VALUE AND ISING TRANSITION

For convenience of the reader we recall the definition of Lyapunov dimension (see, e.g., ref. 20). Let $\{L_j\}_{j=1}^{N}$ denote the Lyapunov exponents of the system in decreasing order, and we assume that $L_j \neq 0$ for all j. The Lyapunov dimension d_ℓ is set equal to 0 if $L_j < 0$, $j = 1,..., N$. Otherwise we consider the quantities

$$R_k = k + \frac{L_1 + \cdots + L_k}{|L_{k+1}|}, \qquad k = 1,..., N-1, \quad R_N = N$$

set $M = \max\{j : L_1 + \cdots + L_j \geqslant 0\}$, and define the Lyapunov dimension as $d_\ell = R_M$. Clearly $d_\ell \in [M, M+1)$.

In Fig. 3 we report the plot of the Lyapunov dimension vs ε for $N = 600$. As we argued in ref. 14, the behavior shown in Fig. 3 indicates that an Ising-type transition can appear for $\varepsilon > 0.55$.

We have been looking for evidence of an Ising-type transitions by looking for possible divergences of the mean square magnetization in the

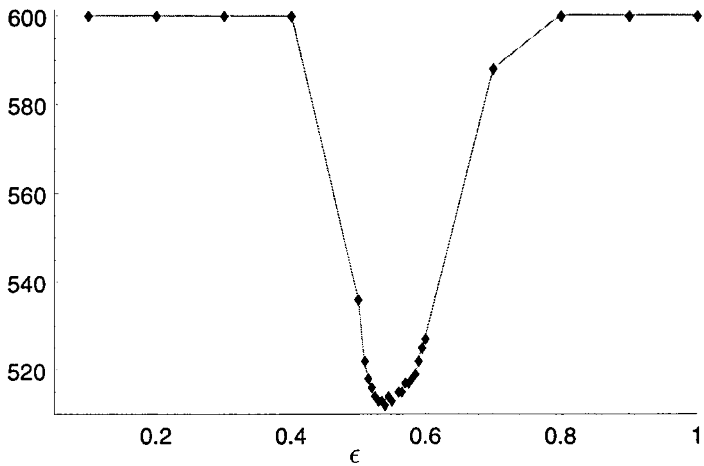

Fig. 3. Lyapunov dimension vs ε. $N = 600$.

stationary state as $\varepsilon \uparrow \varepsilon_c$. The mean square magnetization is defined as the limit as $N \to \infty$ of the quantity

$$NM^2(\varepsilon) = \frac{1}{N} \left\langle \left(\sum_{j \in \mathbb{Z}_N} \text{sign } x_j \right)^2 \right\rangle$$

where $\langle \cdot \rangle$ denotes averaging over the asymptotic stationary state. At the critical point and beyond such quantity should diverge with N.

In computer similations we take of course an averaging over a sample consisting of R runs with independent random initial data, after waiting a "sufficiently large" transient time. N should be chosen, of course, large enough so that we are close to the limit.

The main difficulty in simulations comes from the fact that as $\varepsilon \uparrow \varepsilon_c$, N, which is an upper bound for $NM^2(\varepsilon)$ grows, and the transient time for stationary behavior of the sample average also grows, so that one goes quickly to very large computing times. To tackle this difficulty we resorted for large ε to "preparing" the initial state in such a way as to make it "more similar" to the asymptotic stationary state.

Figure 4 shows the behavior of the mean square magnetization near the critical point. Error bars correspond to one standard error. The data in Fig. 4 are taken after a "sufficiently long" transient time, which depends on the given ε, and is estimated on the basis of the behavior in time of the mean square magnetization.

As we see, the data indicate divergence for $\varepsilon_c \approx 0.58$, with critical exponent around 0.9.

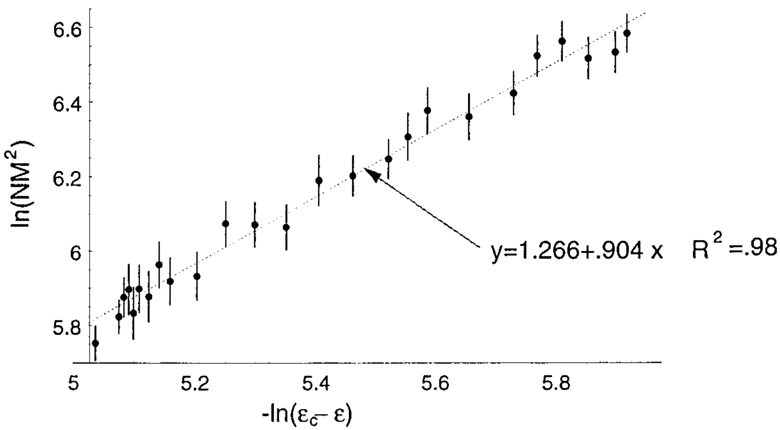

Fig. 4. Log–log plot of the mean square magnetization vs ε.

5. BEHAVIOR FOR LARGE ϵ: TUNNELLING TIME

It is interesting to look at the behavior of the "tunnelling time" as ε grows to the critical point and beyond. Starting from random positive initial data we computed the first time for which the average over all N spins is lower than -0.2 ("tunnelling" or "exit" time). Figure 5 shows the behavior for $N=25$ of the average exit time for a sample of size $R=200$. Error bars correspond to one standard error.

Looking at significantly larger N is made very hard by the enormous size of the exit times, especially near and soon after the critical point, which is made worse by the presence of long tails. We could only find evidence that the exit time plot shown in Fig. 5 is stable for N between 20 and 40.

The plot in Fig. 5 shows that the tunnelling time falls off rather quickly after some critical value $\varepsilon_* > \varepsilon_c$, showing that the diffusive coupling becomes strong enough to drive the system towards a "more cooperative" behavior.

The region of the fall-off of the tunnelling time is already outside the range of Ising-type phenomena. In fact, it is well known[1, 2, 9, 10, 12, 15] that in the range of strong spatial interactions CML's with symmetric diffusion coupling show a strong collective behavior. One of the reasons for that is the symmetry of these systems (see, e.g., ref. 15) which states that any two-one (two-two) space-time periodic solution for some value of ε becomes a two-two (two-one) periodic solution when ε is replaced by $1-\varepsilon$. Thus the region of space interactions which is symmetric with respect to the point $\varepsilon = 1/2$ to the one above the space-time chaos region (see Sections 2 and 3 above) is characterised as well by sustained space-time structures. The

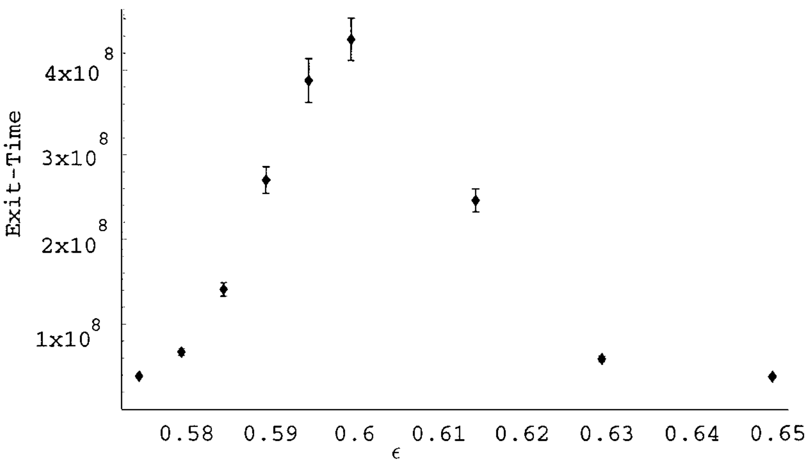

Fig. 5. Tunnelling time vs ε.

region of Ising-type transition (when it exists) should be localized somewhere between these two symmetric regions. It is exactly what has been observed in our computer experiments. It is worth to stress again that this symmetry is caused by the symmetry of the spatial interactions in the considered CML.

ACKNOWLEDGMENTS

C.B., S.F., and A.P. were partially supported by G.N.F.M. and M.U.R.S.T. research funds. L.A.B. was partially supported by NSF Grant DMS-9970215. G.C. was partially supported by M.U.R.S.T. research funds.

REFERENCES

1. H. Chat and M. Courbage (eds.), Lattice dynamics, *Physica D* **103** (1997).
2. K. Kaneko (ed.), Focus issue on coupled map lattices, *Chaos* **2** (1992).
3. L. A. Bunimovich and Ya. G. Sinai, Space-time chaos in coupled map lattices, *Nonlinearity* **1**:491–516 (1988).
4. V. Baladi, M. Degli Esposti, S. Isola, E. Järvenpää, and A. Kupiainen, The spectrum of weakly coupled map lattices, *J. de Mathématiques Pures et Appliquées* **77**:539–584 (1998).
5. J. Bricmont and A. Kupiainen, Coupled analytic maps, *Nonlinearity* **8**:379–396 (1995).
6. M. Jiang and Ya. B. Pesin, Equilibrium measures for coupled map lattices: existence, uniqueness and finite dimensional approximations, *Comm. Math. Phys.* **193**:677–711 (1998).
7. Ya. B. Pesin and Ya. G. Sinai, Space-time chaos in chains of weakly interacting hyperbolic maps, *Adv. Sov. Math.* **3**:165–198 (1991).
8. L. A. Bunimovich and Ya. G. Sinai, Statistical mechanics of coupled map lattices, in *Theory and Applications of Coupled Map Lattices*, K. Kaneko, ed. (Wiley, New York, 1993).
9. L. A. Bunimovich, A. Lambert, and R. Lima, The emergence of coherent structures in coupled map lattices, *J. Stat. Phys.* **61**:253–262 (1990).
10. P. Grassberger and T. Schreiber, Phase transitions in coupled map lattices, *Physica D* **50**:171–188 (1991).
11. S. Sakaguchi, Phase transitions in coupled Bernoulli maps, *Prog. Theor. Phys.* **80**:7–13 (1992).
12. H. Chat and P. Manneville, Collective behavior in spatially extended systems with local interactions and synchronous updating, *Progr. Theor. Phys.* **87**:1–60 (1992).
13. J. Miller and D. A. Huse, Macroscopic equilibrium from microscopic equilibrium from microscopic irreversibility in a chaotic coupled map lattice, *Phys. Rev. E* **48**:2528–2535 (1993).
14. C. Boldrighini, L. A. Bunimovich, G. Cosimi, S. Frigio, and A. Pellegrinotti, Ising-type transitions in coupled map lattices, *J. Stat. Phys.* **80**:1185–1205 (1995).
15. L. A. Bunimovich and E. Carlen, On the problem of stability in lattice dynamical systems, *J. Differential Equations* **123**:213–229 (1995).
16. P. Marcqu and H. Chat, Early-time critical dynamics in lattices of coupled chaotic maps, *Phys. Rev. E* **57**:1591–1603 (1998).

17. G. Gielis and R. S. MacKay, Collective behavior in spatially extended deterministic dynamics. Preprint (Cambridge University, 1999).
18. Ya. G. Sinai, Gibbs measures in ergodic theory, *Russ. Math. Surv.* **28**:21–64 (1972).
19. D. Ruelle, *Thermodynamic Formalism* (Addison–Wesley, 1978).
20. P. Frederickson, J. L. Kaplan, E. D. Yorke, and J. A. Yorke, The Liapunov dimension of strange attractors, *J. Differential Equations* **49**:185–207 (1983).